New Method That Solves the Three-Point Resection Problem Using Straight Lines Intersection

Josep M. Font-Llagunes[1] and Joaquim A. Batlle[2]

Abstract: The three-point resection problem, i.e., the problem of obtaining the position of an unknown point from relative angular measurements to three known stations is a basic operation in surveying engineering. Several approaches to solve this problem, graphically or analytically, have been developed in the last centuries. In this paper, a new analytical approach to solve this problem is presented. The method determines the coordinates of the unknown point by intersecting straight lines through the three stations. The required azimuths of these lines are obtained from the geometric relationship between two similar triangles. Numerical simulations that show the good performance and accuracy of this approach are also reported.

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Introduction

The three-point resection problem, i.e., the problem of obtaining the position of an unknown point from relative angular measurements to three known points (or stations) is a basic operation in *surveying engineering* (Allan et al. 1968).

This problem is also interesting to other scientific fields like *mobile robotics*. In this case, the bearing angles between the lines from a robot point to three known landmarks in the environment are used to determine the robot localization. In robotics this problem is better known as *triangulation*, and studies of it can be found in the following papers (Batlle et al. 2004; Betke and Gurvits 1997; Briechle and Hanebeck 2004; Cohen and Koss 1992; Font-Llagunes and Batlle 2006; Kelly 2003, Piaggio et al. 2001).

The three-point resection problem can be formally defined as follows. Given the two-dimensional coordinates (x_A, y_A), (x_B, y_B), (x_C, y_C) of three known stations A, B, and C, the problem consists in determining the coordinates (x, y) of the unknown point P, from the bearing angles α and β between the lines connecting P and the stations (Fig. 1).

In 1959, the number of different procedures for solving the three-point resection problem was reported to exceed 500 (Bock 1959). However, it is worth noting that these procedures were designed before the computer advent. Hence, most of them are graphical in nature or numerically adapted to be used with the aid of tables. Descriptions of the most relevant ones are typically provided in surveying textbooks such as Allan et al. (1968).

In the last decades, several analytical approaches to solve the three-point resection problem have been formulated. For example the Kaestner–Burkhardt method (Burtch 2005), also referred to as the Pothonot–Snellius method in Allan et al. (1968), the Collins method (Burtch 2005; Klinkenberg 1955), the Cassini method (Burtch 2005; Klinkenberg 1955), or the procedure presented in Danial (1978). In all of these approaches, the position of P is determined in *Cartesian coordinates* from different geometrical relationships between the angles and the position of the four points involved (A, B, C, and P).

A different approach to solve the three-point resection problem is the Tienstra method, also known as the barycentric method (Allan et al. 1968; Hu and Kuang 1997, 1998; Greulich 1999). This method obtains the solution for the position of P in terms of *barycentric coordinates*, first defined in Möbius (1827), which determine the position of P as a linear combination of the stations coordinates. According to Greulich (1999), the earliest recorded occurrence of this method appears to be in Neuberg and Gob (1889).

All the methods above present the singularity of being undetermined if the unknown point P lies on the circle defined by the three known stations A, B, and C; as in this case there are infinite solutions to the problem. Furthermore, if the geometry is close to this, the solution is weak, which means that a small error in the angles will cause a large error in the calculated position. Besides this singularity, the Tienstra method is also undetermined when the stations are aligned, since in this case the position of P cannot be reached using barycentric coordinates, i.e., it cannot be expressed as a linear combination of the stations coordinates.

In this paper, a new analytical method to solve the three-point resection problem is presented. The method is based on the intersection of the straight lines associated with each of the stations to determine the position of P. The required absolute orientation (or azimuth) of these lines is obtained, as it will be explained in further sections, from the geometric relationship between two similar triangles.

This paper is structured as follows. First, in "Three-Point Resection Using Straight Lines Intersection" the proposed approach to solve the three-point resection problem is presented. After this, in "Singularities of Method," the two types of singularities that can appear when using the method are analyzed. In "Steps of

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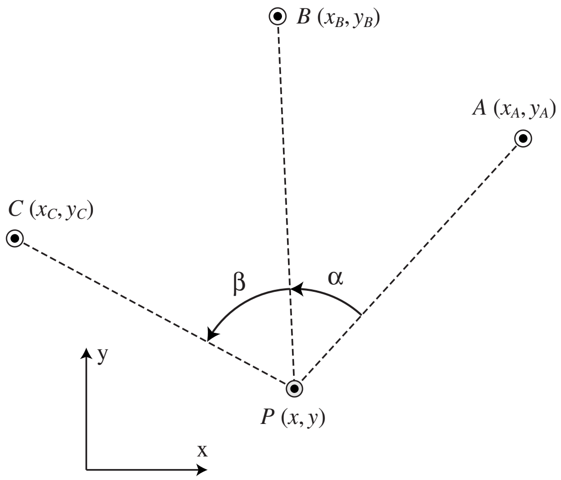

Fig. 1. Three-point resection problem consists of obtaining coordinates of unknown point P from angles α and β, and coordinates of three known stations A, B, and C

Algorithm," the steps that must be followed to apply the method are provided. Next, "Numerical Simulation" reports numerical results that demonstrate the good performance of the presented solution to the three-point resection problem. A realistic surveying engineering example is included in this section. Finally, "Conclusions" summarizes the main points of this paper.

Three-Point Resection Using Straight Lines Intersection

In order to determine the position of P using line intersection, the azimuth—which is the clockwise angle with respect to the north or y direction—of a minimum of two lines from P to the stations (A, B, and C) must be known.

Let's imagine that the azimuth θ of the line from P to A is known (Fig. 2). In this case, the calculus of the position of P is straightforward because the azimuths of the lines PB and PC are, respectively, $\theta-\alpha$ and $\theta-\alpha-\beta$, and therefore, its coordinates can be obtained from the intersection of any pair of the three lines in Fig. 2. Note that the three lines are well defined, because a point of them (A, B, and C) and their absolute orientation (θ, $\theta-\alpha$ and $\theta-\alpha-\beta$), respectively, are known.

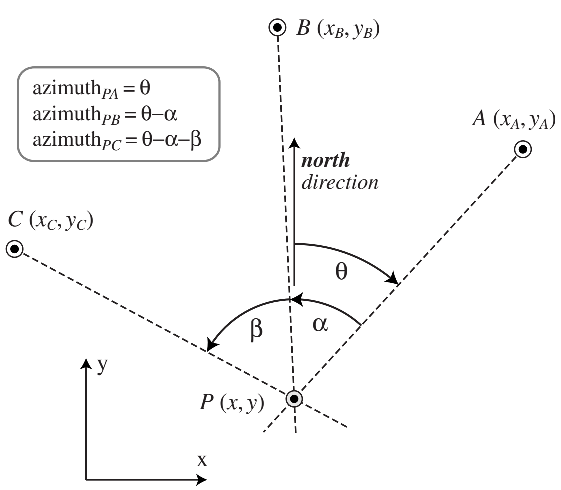

Fig. 2. If azimuth θ is known, position of P can be calculated intersecting any pair of lines through A, B, and C, with absolute angles θ, $\theta-\alpha$, and $\theta-\alpha-\beta$, respectively

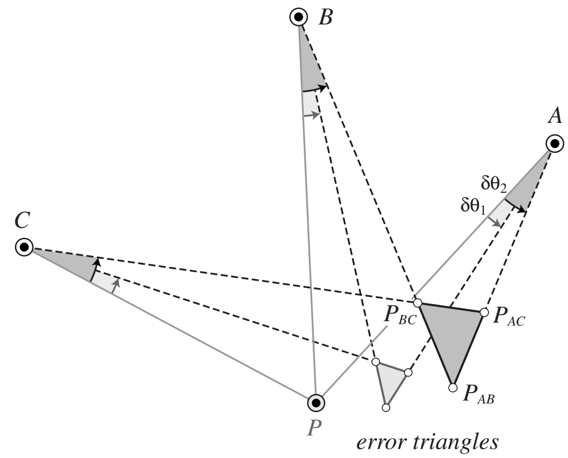

Fig. 3. Use of approximate orientation $\tilde{\theta}=\theta-\delta\theta$ causes three lines to intersect at three points (P_{AB}, P_{BC}, and P_{AC}) which define an *error triangle*. As can be seen, triangle grows as $\delta\theta$ increases.

Nevertheless, in the real application, the actual azimuth θ is not known in advance since the position of P is the unknown of the three-point problem. The proposed method starts using a "nonaccurate" approximation of this azimuth, which we call $\tilde{\theta}$. Obviously, there might be an error $\delta\theta$ between the actual value of θ and the approximation used, $\delta\theta=\theta-\tilde{\theta}$, but as it will be seen this error can be corrected using a geometrical procedure.

The next step consists of determining the points at which the lines through A, B, and C with azimuths $\tilde{\theta}$, $\tilde{\theta}-\alpha$, and $\tilde{\theta}-\alpha-\beta$ intersect. Note that in this case, due to the error $\delta\theta$, the lines intersect in three different points P_{AB}, P_{BC}, and P_{AC} (Fig. 3) instead of one (Fig. 2). From now on, the triangle defined by these points will be called an *error triangle*. The coordinates of points P_{AB}, P_{BC}, and P_{AC} can be obtained by means of the following expressions:

$$x_{P_{AB}}=\frac{m_A x_A - m_B x_B - y_A + y_B}{m_A - m_B}, \quad y_{P_{AB}}=m_A(x_{P_{AB}}-x_A)+y_A \tag{1}$$

$$x_{P_{BC}}=\frac{m_B x_B - m_C x_C - y_B + y_C}{m_B - m_C}, \quad y_{P_{BC}}=m_B(x_{P_{BC}}-x_B)+y_B \tag{2}$$

$$x_{P_{AC}}=\frac{m_A x_A - m_C x_C - y_A + y_C}{m_A - m_C}, \quad y_{P_{AC}}=m_A(x_{P_{AC}}-x_A)+y_A \tag{3}$$

where m_A, m_B, and m_C=slopes of the lines through A, B, and C, which can be written in terms of the cotangent of the azimuths using

$$m_A=\cot(\tilde{\theta}); \quad m_B=\cot(\tilde{\theta}-\alpha); \quad m_C=\cot(\tilde{\theta}-\alpha-\beta) \tag{4}$$

It can be noticed from Fig. 3, that the bigger the error $\delta\theta$ is, the bigger the area of the triangle defined by the three intersection points is. This means that the azimuth error $\delta\theta$ and the geometry of the error triangle are related. It would be highly valuable to

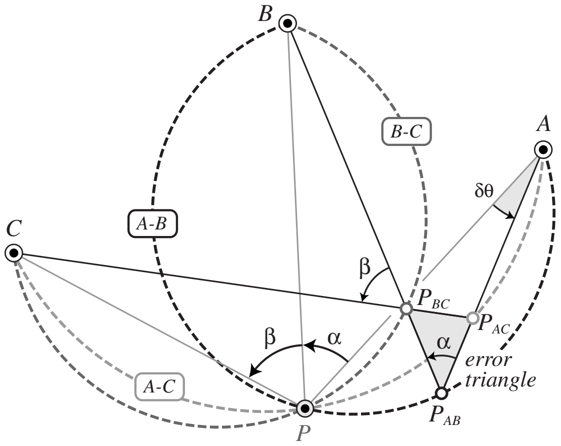

Fig. 4. Circumferences through stations A, B, and C, and vertexes of error triangle

have such geometric relationship in *analytical form*, because then the actual azimuth from P to A could be calculated, $\theta=\tilde{\theta}+\delta\theta$, and therefore the position of P obtained by straight lines intersection (see Fig. 2). In this paper, we prove that this analytical relationship exists, and we derive the formulation to recover the actual value of θ.

According to Fig. 4, it can be seen that the vertex P_{AB} belongs to the circumference that also contains points A, B, and P (unknown) which is well defined because three of their points are known (A, B, and P_{AB}). The same reasoning applies to points P_{BC} and P_{AC} which belong to the circumferences shown in the figure. Note that the angles between the sides of the error triangle (which are the intersected straight lines) are the measured angles α and β, and $\pi-\alpha-\beta$.

Fig. 5 shows the centers of the circumferences in Fig. 4. In Fig. 5 C_{AB} refers to the center of the circumference through A, B, and P, and the same reasoning applies to points C_{BC} and C_{AC}. The coordinates of these points can be found from the coordinates of A, B, and C, and the measured angles α and β using the following equations (Cohen and Koss 1992):

$$\begin{pmatrix} x_{C_{AB}} \\ y_{C_{AB}} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} x_A + x_B + (y_A - y_B)\cot\alpha \\ y_A + y_B + (x_B - x_A)\cot\alpha \end{pmatrix} \tag{5}$$

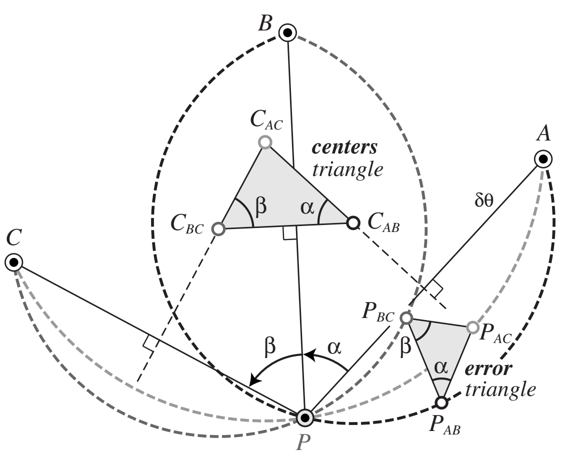

Fig. 5. Error triangle and centers triangle are similar because both have same three angles between their sides

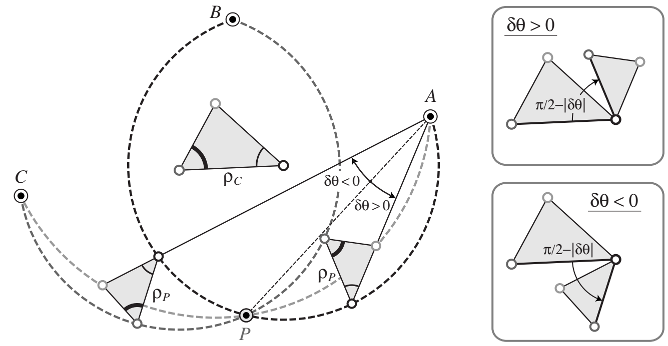

Fig. 6. Relationship between sign of $\delta\theta$ and relative orientation between error and centers triangles

$$\begin{pmatrix} x_{C_{BC}} \\ y_{C_{BC}} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} x_B + x_C + (y_B - y_C)\cot\beta \\ y_B + y_C + (x_C - x_B)\cot\beta \end{pmatrix} \tag{6}$$

$$\begin{pmatrix} x_{C_{AC}} \\ y_{C_{AC}} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} x_A + x_C + (y_A - y_C)\cot(\alpha+\beta) \\ y_A + y_C + (x_C - x_A)\cot(\alpha+\beta) \end{pmatrix} \tag{7}$$

From now on, the triangle defined by the points C_{AB}, C_{BC}, and C_{AC} will be called *centers triangle*. The centers triangle is geometrically obtained by the intersection of the perpendicular bisectors of the segments PA, PB, and PC, and therefore the angle between the sides of the triangle are the measured angles α and β, and $\pi-\alpha-\beta$ (Fig. 5). This means that the error triangle and the centers triangle are *similar* because both have the same three angles between their sides.

Note that the centers triangle is a fixed triangle uniquely defined by the inputs of the problem (i.e., the coordinates of A, B, and C, and the angles α and β), and conversely the error triangle shape depends upon the error $\delta\theta$ in the initial guess $\tilde{\theta}$ (Fig. 3).

Taking into account the geometry implied in the problem, it can be demonstrated that the *similarity ratio* r, which represents the scale factor between two corresponding sides of both triangles, depends upon $\delta\theta$ in the following analytical form:

$$r = \frac{\rho_P}{\rho_C} = 2\sin|\delta\theta| \tag{8}$$

where ρ_P and ρ_C stand for the length of corresponding sides of the error triangle and the centers triangle, respectively (Fig. 6), for example $\rho_P=\|P_{AB}P_{BC}\|$ and $\rho_C=\|C_{AB}C_{BC}\|$ (but any other corresponding sides can be used). In the previous expressions, $\|\mathbf{x}\|$ denotes the Euclidean norm (or length) of the vector $\mathbf{x}$. From Eq. (8), the following expression can be used to determine the absolute value of $\delta\theta$:

$$|\delta\theta| = \arcsin\left(\frac{\rho_P}{2\rho_C}\right) \tag{9}$$

Note that the similarity ratio r also relates the areas of both triangles in the form $r=\sqrt{A_P/A_C}$, where A_P and A_C stand for the areas of the error triangle, and the centers triangle, respectively. Then, we obtain an equation similar to Eq. (9) but using areas

$$|\delta\theta| = \arcsin\left(\frac{1}{2}\sqrt{\frac{A_P}{A_C}}\right) \tag{10}$$

where A_P and A_C can be calculated using the following matrix determinants:

$$A_P = \pm \frac{1}{2}\begin{vmatrix} x_{P_{AB}} & x_{P_{BC}} & x_{P_{AC}} \\ y_{P_{AB}} & y_{P_{BC}} & y_{P_{AC}} \\ 1 & 1 & 1 \end{vmatrix}, \quad A_C = \pm \frac{1}{2}\begin{vmatrix} x_{C_{AB}} & x_{C_{BC}} & x_{C_{AC}} \\ y_{C_{AB}} & y_{C_{BC}} & y_{C_{AC}} \\ 1 & 1 & 1 \end{vmatrix} \tag{11}$$

where the signs are chosen in order to make the areas positive.

Eqs. (9) and (10) are useful to determine the magnitude of the error in $\tilde{\theta}$, however they do not specify the sign of it. The sign can be determined from the relative orientation of the error triangle with respect to the centers triangle. As it can be seen in Fig. 6, the orientation of one side of the error triangle with respect to the corresponding one in the centers triangle is $\pi/2-|\delta\theta|$ *clockwise* if $\delta\theta>0$, and the same magnitude *counterclockwise* if $\delta\theta<0$. Therefore, the sign of $\delta\theta$ coincides with the sign of the z component of the cross product of one side of the error triangle with the corresponding side of the centers triangle

$$\begin{aligned} \mathrm{sign}(\delta\theta) &= \mathrm{sign}([P_{ij}P_{jk} \times C_{ij}C_{jk}]_z) \\ &= \mathrm{sign}((x_{P_{jk}} - x_{P_{ij}})(y_{C_{jk}} - y_{C_{ij}}) - (x_{C_{jk}} - x_{C_{ij}})(y_{P_{jk}} - y_{P_{ij}})) \end{aligned} \tag{12}$$

where $i,j,k=\{A,B,C\}$, $i\neq j\neq k$.

Once the error is known in magnitude, using either Eq. (9) or (10), and sign, Eq. (12), the actual azimuth of the line from P to A can be found using $\theta=\tilde{\theta}+\delta\theta$. Finally, any pair of the lines through A, B, and C with azimuths θ, $\theta-\alpha$, and $\theta-\alpha-\beta$ can be intersected in order to calculate the coordinates of point P, which is the goal of the *three-point resection problem*.

It is worth noting that the initial approximate value of the azimuth from P to A ($\tilde{\theta}$) does not need to be very accurate, since the provided algorithm can correct any error in this approximation within the interval $[-\pi/2 \text{ rad}, \pi/2 \text{ rad}]$.

Singularities of Method

The singularities of the method arise either when P is aligned with two of the stations used (A,B,C), because both triangles have an undetermined vertex, or when P lies on the circumference that define the three stations, because both triangles reduce to a point. While the first singularity is due to the formulation used in the method, the second is intrinsic to the three-point resection problem and therefore all the methods to solve this problem suffer from it. In what follows, these two singularities will be analyzed.

Point P Aligned with Two of Stations

This singularity is easy to detect because one of the angles α, β, or $\alpha+\beta$ is equal to 0 or π rad. In this case, the vertex of the error triangle associated with the two stations with which P is aligned, cannot be calculated because the two corresponding lines are parallel. Besides this, the corresponding circumference degenerates to a line, and neither can its center be calculated. Therefore, in this case, both triangles have only one finite side. Fig. 7 illustrates the particular case when P is aligned with stations A and B ($\alpha=0$ rad), note that neither P_{AB} nor C_{AB} can be determined.

However, this singularity does not represent a problem for the method presented because the finite sides of the triangles can be used in Eqs. (9) and (12) in order to determine the magnitude and sign of the azimuth error $\delta\theta$. Once this error is known, the coordinates of P can be calculated by line intersection using any pair of stations except from the one with which P is aligned. In the case of Fig. 7, either the pair B-C or the pair A-C could be used to determine the coordinates of P.

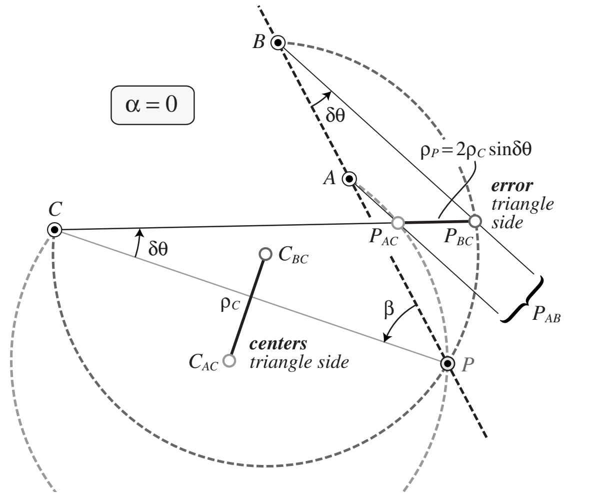

Fig. 7. Singularity that arises when P is aligned with two of stations, representing particular case when these stations are A and B, $\alpha=0$ rad

Point P on Circumference Defined by Stations

As noted before this singularity is intrinsic to the three-point resection problem, and makes the problem undetermined. As it can be seen in Fig. 8, the two triangles reduce to a point for any value of the error $\delta\theta$. Therefore, this error cannot be determined and neither can the exact position of point P.

This singularity can also be easily detected, because it can be demonstrated that in this case $\alpha+\beta+\hat{B}=\pi$ rad, $\hat{B}\in[0,\pi \text{ rad})$ being the angle between the lines AB and BC (Fig. 8). This angle can be calculated from the coordinates of the stations using the law of cosines

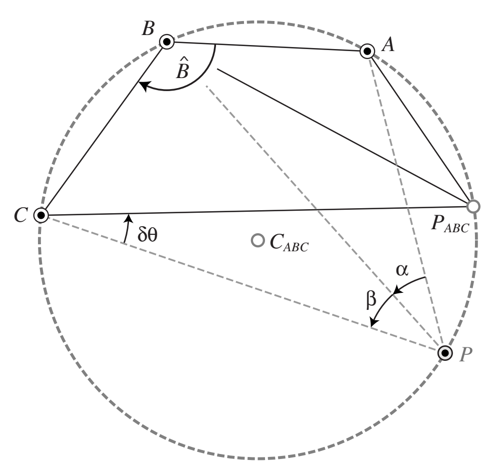

Fig. 8. Singularity that arises when P lies on circumference defined by stations A, B, and C

Table 1. Steps of Presented Algorithm to Solve Three-Point Resection Problem

1. Compute $\hat{B}$ from Eq. (13).
2. If $\alpha+\beta+\hat{B}=\pi$ rad$\rightarrow$ calculus of $\delta\theta$ is *undetermined*. Use the best estimation of θ, if available, and go to Step 10
   If $\alpha+\beta+\hat{B}\neq\pi$ rad$\rightarrow$ go to Step 3.
3. Provide an approximation $\tilde{\theta}$ of the azimuth of the line from P to A.
4. Define the lines through A, B, and C, with azimuths $\tilde{\theta}$, $\tilde{\theta}-\alpha$ and $\tilde{\theta}-\alpha-\beta$.
5. Compute the coordinates of the vertexes of the error triangle P_{AB}, P_{BC}, and P_{AC} intersecting the lines defined in Step 4. Use Eqs. (1)–(3).
6. Compute the coordinates of the vertexes of the centers triangle C_{AB}, C_{BC}, and C_{AC} using Eqs. (5)–(7).
7. Compute the magnitude $|\delta\theta|$ of the error in $\tilde{\theta}$ using either Eq. (9) or Eq. (10).
8. Compute the sign of the error, sign($\delta\theta$), using Eq. (12).
9. Compute the real orientation of line PA using: $\theta=\tilde{\theta}+\text{sign}(\delta\theta)\cdot|\delta\theta|$.
10. Determine the coordinates (x,y) of P intersecting any pair of the three lines through A, B, and C, with azimuths θ, $\theta-\alpha$, and $\theta-\alpha-\beta$.
    Use Eqs. (1), (2) or (3) with the corrected value of the azimuths.

$$\rho_{AC}^2=\rho_{AB}^2+\rho_{BC}^2-2\rho_{AB}\rho_{BC}\cos\hat{B} \tag{13}$$

where $\rho_{ij}=\sqrt{(x_j-x_i)^2+(y_j-y_i)^2}$; $(i,j=\{A,B,C\},i\neq j)$=Euclidean distance between any two stations i and j.

In spite of this singularity, the position of P can be determined provided that the azimuths of the lines θ, $\theta-\alpha$, and $\theta-\alpha-\beta$ are known; or it can be estimated using the best estimation available of the above angles. In this case, the accuracy of the position estimation will depend upon the accuracy of the absolute angles used.

Steps of Algorithm

This section describes the steps of the algorithm application. The inputs of the algorithm are the coordinates of the stations (x_A,y_A), (x_B,y_B), and (x_C,y_C), and the measured angles between stations α and β (Fig. 1). From these data, the proposed algorithm determines the coordinates of P following the steps in Table 1.

As it has been pointed out in "Singularities of Method," if point P is aligned with any two stations, namely i and j—the other station is called k $(i,j,k=\{A,B,C\},i\neq j\neq k)$—then the vertexes P_{ij} and C_{ij} of the error and centers triangles cannot be calculated. Nevertheless, the magnitude and sign of $\delta\theta$ (Steps 7 and 8 of the algorithm) can be found from Eqs. (9) and (12) using the finite sides of both triangles: $P_{ik}P_{jk}$ and $C_{ik}C_{jk}$. Note that in this case, since one vertex of each triangle is missing, the relation of areas in Eq. (10) cannot be used to determine $|\delta\theta|$.

Numerical Simulation

In order to validate the performance of the proposed algorithm, two numerical simulations and a realistic surveying engineering example have been carried out using the mathematical software MATLAB R2007a. In the two simulations, the three stations form an equilateral triangle inscribed in a circle of 10 m radius and centered at the origin of the coordinate system. The coordinates of the stations are given in Table 2. Note that with this location of the stations, from Eq. (13) we obtain $\hat{B}=\pi/3$ rad.

Table 2. Coordinates of Stations A, B, and C in Numerical Simulations

Station A	Station B	Station C
$x_A=0$ m	$x_B=-5\sqrt{3}$ m	$x_C=5\sqrt{3}$ m
$y_A=10$ m	$y_B=-5$ m	$y_C=-5$ m

Table 3 shows the input data in both numerical simulations. As it can be seen, in the second simulation P is aligned with stations B and C, since $\beta=\pi$ rad. Note that the coordinates of P at each simulation are also given in the table. This is the result at which the presented algorithm has to arrive. Fig. 9 shows the position of the stations A, B, and C, and the two points P in each simulation (P_1 and P_2). In both cases $\alpha+\beta+\hat{B}\neq\pi$ rad, i.e., the points to be reached do not lie on the circumference defined by the three stations. Therefore, the next two subsections will begin in Step 3 of the algorithm presented in Table 1.

Numerical Simulation 1

In Step 3 (Table 1) we provide the following approximation of the azimuth of the line from P to A: $\tilde{\theta}=6.2$ rad. We start with this value because, even without knowing the exact location of P, from the layout of the involved points it can be noticed that the azimuth θ is a bit lower than $2\pi\approx 6.28$ rad. Nevertheless, as it has been mentioned before, the approximation $\tilde{\theta}$ does not need to be very accurate.

Then, defining the three lines in Step 4 and intersecting them, Step 5, the following vertexes of the *error triangle* are obtained

$$\begin{pmatrix} x_{P_{AB}} \\ y_{P_{AB}} \end{pmatrix}=\begin{pmatrix} 0.8956 \\ -0.7413 \end{pmatrix}\text{ m},\quad \begin{pmatrix} x_{P_{BC}} \\ y_{P_{BC}} \end{pmatrix}=\begin{pmatrix} 4.6217 \\ 0.9192 \end{pmatrix}\text{ m},$$
$$\begin{pmatrix} x_{P_{AC}} \\ y_{P_{AC}} \end{pmatrix}=\begin{pmatrix} 0.2191 \\ 7.3718 \end{pmatrix}\text{ m} \tag{14}$$

Following, in Step 6, we determine the coordinates of the *centers triangle* vertexes using Eqs. (5)–(7)

$$\begin{pmatrix} x_{C_{AB}} \\ y_{C_{AB}} \end{pmatrix}=\begin{pmatrix} -6.9499 \\ 4.0125 \end{pmatrix}\text{ m},\quad \begin{pmatrix} x_{C_{BC}} \\ y_{C_{BC}} \end{pmatrix}=\begin{pmatrix} 0 \\ -6.5714 \end{pmatrix}\text{ m},$$
$$\begin{pmatrix} x_{C_{AC}} \\ y_{C_{AC}} \end{pmatrix}=\begin{pmatrix} 17.5653 \\ 10.1413 \end{pmatrix}\text{ m} \tag{15}$$

Table 3. Input Data for Simulations 1 and 2

Data	Simulation 1	Simulation 2
Coordinates of P (m)	(2,2)	(2,−5)
Angle α (rad)	1.9068	1.4382
Angle β (rad)	1.7503	3.1416

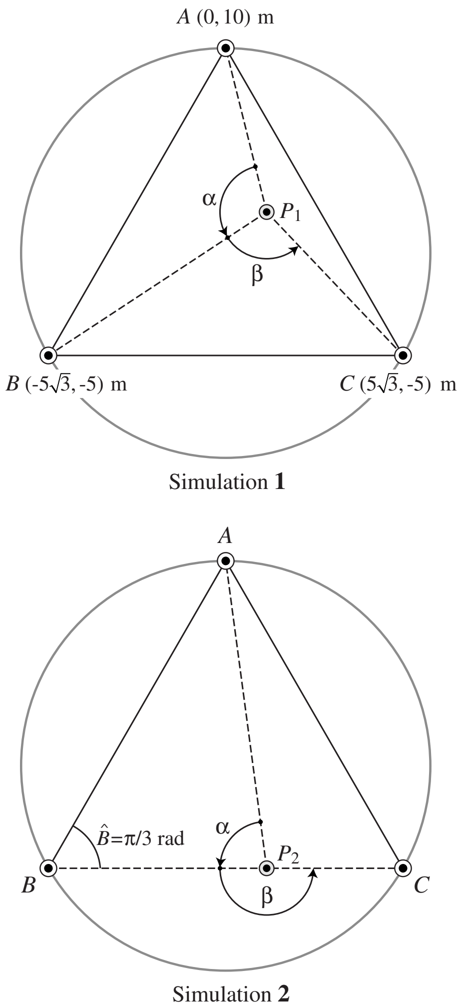

Fig. 9. Layout and coordinates of stations in numerical simulations, showing position of point P in each simulation (P_1 and P_2), and angles α and β

Once both *similar* triangles are defined, the magnitude of the error $\delta\theta$ can be determined using either Eq. (9) or (10), and its sign by means of Eq. (12), Steps 7 and 8 of the algorithm

$$|\delta\theta| = 0.1618 \text{ rad}, \quad \text{sign}(\delta\theta) < 0 \tag{16}$$

which yields the following actual value of the azimuth θ (Step 9)

$$\theta = \tilde{\theta} + \delta\theta = 6.2 - 0.1618 = 6.0382 \text{ rad} \tag{17}$$

Finally, intersecting any pair of the three lines defined in Step 10 using the value of θ above, Eq. (17), the following coordinates of P are obtained:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2.0000 \\ 2.0000 \end{pmatrix} \text{ m} \tag{18}$$

which is the actual position of P. It is worth noting that the numerical accuracy in the calculations is the maximum provided by MATLAB, i.e., 10^{-16}.

Numerical Simulation 2

In this case, in the Step 3 of the algorithm, we provide the same approximation for the azimuth from P to A: $\tilde{\theta}=6.2$ rad. Then, the three lines in Step 4 are defined. Note that the lines through B and C are parallel (because P is aligned with B and C, i.e., $\beta=\pi$ rad) and therefore only the coordinates of the vertexes P_{AB} and P_{AC} can be calculated in Step 5

$$\begin{pmatrix} x_{P_{AB}} \\ y_{P_{AB}} \end{pmatrix} = \begin{pmatrix} 1.2917 \\ -5.4917 \end{pmatrix} \text{ m}, \quad \begin{pmatrix} x_{P_{AC}} \\ y_{P_{AC}} \end{pmatrix} = \begin{pmatrix} 1.2200 \\ -4.6324 \end{pmatrix} \text{ m} \tag{19}$$

In Step 6 we determine the coordinates of the points C_{AB} and C_{AC} that define the finite side of the centers triangle using Eqs. (5) and (7)

$$\begin{pmatrix} x_{C_{AB}} \\ y_{C_{AB}} \end{pmatrix} = \begin{pmatrix} -3.3301 \\ 1.9226 \end{pmatrix} \text{ m}, \quad \begin{pmatrix} x_{C_{AC}} \\ y_{C_{AC}} \end{pmatrix} = \begin{pmatrix} 5.3301 \\ 3.0774 \end{pmatrix} \text{ m} \tag{20}$$

The point C_{BC} cannot be reached because $\beta=\pi$ rad and Eq. (6) is undetermined since $\cot\beta \to \infty$. Once both finite sides of the triangles are defined, the magnitude of the error $\delta\theta$ and its sign are determined using the following expressions:

$$|\delta\theta| = \arcsin\left(\frac{\sqrt{(x_{P_{AC}} - x_{P_{AB}})^2 + (y_{P_{AC}} - y_{P_{AB}})^2}}{2\sqrt{(x_{C_{AC}} - x_{C_{AB}})^2 + (y_{C_{AC}} - y_{C_{AB}})^2}} \right) = 0.0494 \text{ rad} \tag{21}$$

$$\text{sign}(\delta\theta) = \text{sign}([P_{AB}P_{AC} \times C_{AB}C_{AC}]_z) < 0 \tag{22}$$

Note that in this case the relationship between areas, Eq. (10), cannot be used to determine $|\delta\theta|$ because they are not defined. The results above yield the following actual value of the azimuth θ (Step 9):

$$\theta = \tilde{\theta} + \delta\theta = 6.2 - 0.0494 = 6.1506 \text{ rad} \tag{23}$$

Finally, in Step 10, the three lines through the stations are defined, but lines through B and C are parallel because P is aligned with them. Therefore, the coordinates of the unknown point P can be obtained intersecting the line through A either with the line through B or the line through C. In both cases the same coordinates are obtained

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2.0000 \\ -5.0000 \end{pmatrix} \text{ m} \tag{24}$$

which is the actual position of P for the second simulation.

Surveying Engineering Example

Table 4. Coordinates of Stations A, B, and C in Surveying Engineering Example

Station A	Station B	Station C
x_A=5,297.154 m	x_B=4,905.726 m	x_C=4,908.975 m
y_A=7,050.825 m	y_B=7,221.493 m	y_C=7,658.629 m

The coordinates of the stations in this example are given in Table 4. With this location of the stations, from Eq. (13) we obtain $\hat{B}=1.97453$ rad. In this case, the input angular measurements are $\alpha=0.70842$ rad and $\beta=0.16247$ rad. Then, we obtain $\alpha+\beta+\hat{B}=2.8454 \neq \pi$ rad, Step 2 of the algorithm, which indicates that we can proceed to Step 3 (i.e., the solution of the problem can be determined).

As a first approximation of the azimuth from P to A, we use $\tilde{\theta}=\pi/4$ rad. Then, we obtain the following coordinates of the vertexes of the *error triangle* (Steps 4 and 5):

$$\begin{pmatrix} x_{P_{AB}} \\ y_{P_{AB}} \end{pmatrix} = \begin{pmatrix} 4{,}858.748 \\ 6{,}612.419 \end{pmatrix} \text{ m}, \quad \begin{pmatrix} x_{P_{BC}} \\ y_{P_{BC}} \end{pmatrix} = \begin{pmatrix} 4{,}925.011 \\ 7{,}471.520 \end{pmatrix} \text{ m},$$

$$\begin{pmatrix} x_{P_{AC}} \\ y_{P_{AC}} \end{pmatrix} = \begin{pmatrix} 4{,}987.596 \\ 6{,}741.267 \end{pmatrix} \text{ m} \tag{25}$$

and the *centers triangle* (Step 6)

$$\begin{pmatrix} x_{C_{AB}} \\ y_{C_{AB}} \end{pmatrix} = \begin{pmatrix} 5{,}001.842 \\ 6{,}907.731 \end{pmatrix} \text{ m}, \quad \begin{pmatrix} x_{C_{BC}} \\ y_{C_{BC}} \end{pmatrix} = \begin{pmatrix} 3{,}573.949 \\ 7{,}449.971 \end{pmatrix} \text{ m},$$

$$\begin{pmatrix} x_{C_{AC}} \\ y_{C_{AC}} \end{pmatrix} = \begin{pmatrix} 4{,}847.141 \\ 7{,}191.280 \end{pmatrix} \text{ m} \tag{26}$$

Once both *similar* triangles are obtained, as before, the magnitude of $\delta\theta$ can be determined using either Eq. (9) or (10), and its sign by means of Eq. (12) (Steps 7 and 8)

$$|\delta\theta| = 0.28595 \text{ rad}, \quad \text{sign}(\delta\theta) > 0 \tag{27}$$

which yields the following actual value of θ (Step 9):

$$\theta = \tilde{\theta} + \delta\theta = \pi/4 + 0.28595 = 1.07135 \text{ rad} \tag{28}$$

Finally, intersecting any pair of the three lines defined in Step 10, using the value of θ above, the following coordinates of the unknown point P are obtained:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4{,}721.686 \\ 6{,}736.857 \end{pmatrix} \text{ m} \tag{29}$$

which is the actual position of P in the example considered.

Conclusions

In this paper, a new analytical solution to the three-point resection problem has been presented. This method uses the similarity ratio between two triangles, readily determined from the input data, to determine the azimuths of the lines through the stations and the unknown point. Once these azimuths are reached, the coordinates of the unknown point are determined by intersecting any pair of the lines.

This method represents an alternative to the other existing solutions of the three-point resection problem, and its good performance has been proven by means of numerical simulations and a realistic surveying example using MATLAB. One advantage of the method is that all the points involved (vertexes of the triangles and unknown point) are obtained by means of straight line intersection which makes the analytical formulation very simple. Furthermore, the method is only undetermined when the point lies on the circumference defined by the stations, singularity that is intrinsic to the three-point resection problem, and even in this case the position of P can be estimated with an accuracy that depends upon the accuracy of the azimuths estimation. Other approaches suffer from more singularities that make the unknown point unreachable, to name one, the Tienstra method is undetermined when the three stations are aligned.

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